

# Finiteness of prescribed fibers of local biholomorphisms: a geometric approach

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## Abstract

Let  $X$  be a Stein manifold of complex dimension at least two,  $F : X \rightarrow \mathbb{C}^n$  a local biholomorphism, and  $q \in F(X)$ . In this paper we formulate sufficient conditions, involving only objects naturally associated to  $q$ , in order for the fiber over  $q$  to be finite. Assume that  $F^{-1}(l)$  is 1-connected for the generic complex line  $l$  containing  $q$ , and  $F^{-1}(l)$  has finitely many components whenever  $l$  is an exceptional line through  $q$ . Using arguments from topology and differential geometry, we establish a sharp estimate on the size of  $F^{-1}(q)$ . It follows that for  $n \geq 2$  a local biholomorphism of  $X$  onto  $\mathbb{C}^n$  is invertible if and only if the pull-back of every complex line is 1-connected.

## 1 Introduction.

The theory of holomorphic maps  $F : X \rightarrow \mathbb{C}^n$ , where  $X$  is a connected non-compact complex manifold of dimension  $n \geq 2$ , has been the object of much research over the years. As expected, most of these works make extensive use of various analytical tools, a prime example being Nevanlinna theory.

In this paper, by contrast, we use only topological and geometric arguments to address the problem of estimating the size of a fiber, specified in advance, of a given local biholomorphism:

**Theorem 1.1.** *Let  $X$  be a connected non-compact complex manifold of dimension at least two,  $F : X \rightarrow \mathbb{C}^n$  a local biholomorphism, and  $q$  a point in the image of  $F$ . Assume that*

- a)  $X$  carries a complete Kähler metric of negative holomorphic sectional curvature.*
- b) The pre-image of the generic complex line containing  $q$  is 1-connected.*
- c) The pre-image of any exceptional complex line through  $q$  has finitely many components.*

*Then, the fiber  $F^{-1}(q)$  is finite. In fact,*

$$\#F^{-1}(q) \leq \min_{V \subset \mathbb{C}^n} \max_{l \in V} \{\# \text{components of } F^{-1}(l)\}, \quad (1.1)$$

*where the minimum is taken over all complex 2-planes  $V$  containing  $q$ , and the maximum is taken over all complex lines  $l$  in  $V$  that pass through  $q$ .*

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The topological hypotheses b) and c) in Theorem 1.1 are essential for the validity of the theorem, and the estimate (1.1) is sharp (see Section 3). The necessity for the geometric hypothesis a), on the other hand, seems to be a subtler matter, and remains an unsettled question.

The word *generic* in the statement of Theorem 1.1 is, of course, to be understood in the usual sense. For the sake of clarity, and to establish notation, we recall its meaning. After a translation by  $-q$ , the set of complex lines in  $\mathbb{C}^n$  that pass through  $q$  is naturally identified with the complex projective space  $\mathbb{CP}^{n-1}$ . A property  $\mathcal{P}$  about such lines is satisfied generically if there exists a complex hypersurface  $\Sigma \subset \mathbb{CP}^{n-1}$  such that  $\mathcal{P}$  holds for all elements of  $\mathbb{CP}^{n-1} - \Sigma$ . Lines associated to  $\Sigma$  are deemed non-generic or *exceptional*. If  $n = 2$ ,  $\mathcal{P}$  is satisfied generically if it holds for all lines arising from the complement of a finite subset of  $\mathbb{CP}^1$ .

It should be pointed out that the estimate on the size of the fiber  $F^{-1}(q)$  involves only objects naturally associated to  $q$ . We also observe that when  $n = 2$  the actual finite number of exceptional lines in Theorem 1.1 is not relevant for the estimate on the size of the fiber. In this case, (1.1) reduces to

$$\#F^{-1}(q) \leq \max_l \{\#\text{components of } F^{-1}(l)\}, \quad (1.2)$$

where a)-c) are in force, and the maximum is taken over all complex lines  $l$  containing  $q$ .

The theorem is new even in the classical case  $X = \mathbb{C}^n$ ,  $n \geq 2$ . In fact, it holds if  $X$  is a Stein manifold. Indeed, an explicit construction of complete Kähler metrics in  $\mathbb{C}^N$  with negative sectional curvature, hence with negative holomorphic sectional curvature, can be found in [12] (see also [5], [13]). Being Stein,  $X$  can be properly embedded as a complex submanifold of some  $\mathbb{C}^N$ . Endowing the latter with a metric as above, it follows from ([6], p.164, 176) that the induced metric on  $X$  satisfies a) in Theorem 1.1.

**Corollary 1.2.** *Let  $X$  be a connected Stein manifold of complex dimension at least two,  $F : X \rightarrow \mathbb{C}^n$  a local biholomorphism, and  $q$  a point in the image of  $F$ . If  $F^{-1}(l)$  is 1-connected for the generic complex line  $l$  containing  $q$ , and  $F^{-1}(l)$  has finitely many components whenever  $l$  is an exceptional line through  $q$ , then  $F^{-1}(q)$  is finite.*

The next result is a direct consequence of estimate (1.1):

**Corollary 1.3.** *Let  $X$  be a connected complex  $n$ -manifold that carries a complete Kähler metric of negative holomorphic sectional curvature, and  $F : X \rightarrow \mathbb{C}^n$  a local biholomorphism,  $n \geq 2$ . If the pre-image under  $F$  of every affine complex line that intersects  $F(X)$  is 1-connected, then  $F$  is injective.*

**Corollary 1.4.** *A surjective local biholomorphism  $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ ,  $n \geq 2$ , is invertible if and only if the pre-image of every complex line is a 1-connected set.*

The proof of Theorem 1.1 represents a substantial elaboration of the basic idea that motivated the main result in [8]:

*A local biholomorphism  $F : Y \rightarrow \mathbb{C}^n$ ,  $n \geq 2$ , is injective if the pre-image of every complex line that intersects  $F(Y)$  is conformal to a connected rational curve.*

(As communicated to us by S. Nollet, in the special case of certain algebraic maps the above result follows from the famous Bend-and-Break Lemma of S. Mori.) However, we stress that the techniques and results in [8] are quite different from the ones in the present paper.

The starting point of these investigations is the naive observation that a map into  $\mathbb{C}^n$  that has discrete fibers is injective if and only if the pre-image of any given point is a connected set.

The next logical step consists in trying to infer injectivity ([1],[8]) or, in the present case, the finiteness of fibers, from the requirement that the pre-images of one-dimensional objects (such as complex lines), rather than points, are connected.

Thus far injectivity can be achieved if  $X$  carries the right geometry, but only under the stronger assumption that the pull-backs of *all* complex lines that intersect the image are not only connected, but actually 1-connected (Corollary 1.3).

In what follows we elaborate on the relationship between differential geometry and the problem of estimating the size of a fiber of a local biholomorphism. Along the way, we provide a rough outline for the proof of Theorem 1.1.

The pre-image of a generic complex line is a 1-connected two-dimensional real surface that is properly embedded in  $X$  (hence complete relative to the induced metric). Furthermore, it has negative curvature (more generally, the holomorphic sectional curvature of a Kähler submanifold is not greater than that of the ambient manifold ([6], p.164, 176)).

Thus, the pre-image of such a complex line is a Cartan-Hadamard surface, and therefore any two distinct points in it can be joined by a unique geodesic.

After reducing the proof of Theorem 1.1 to the (complex) two-dimensional case, in Theorem 2.1, one can take advantage of the above mentioned uniqueness of geodesics and construct a continuous nowhere vanishing section  $s$  of the restriction of the tautological line bundle  $L$  to the complement of the finite set in  $\mathbb{CP}^1$  that corresponds to the exceptional lines whose pre-images are not 1-connected.

One then proceeds to show – and this is the heart of the matter –, that if the estimate (1.1) does not hold then each one of the singularities of  $s$  must have index zero.

The arguments employed to establish the vanishing of the local indices involve once again the use of geodesics, this time to deform continuously the section  $s$  into another one whose local index at the singularity in question is manifestly zero. By the Poincaré-Hopf theorem, the Euler number  $e(L)$  of  $L$  is the sum of the local indices of all the singularities of the section  $s$ , and so it would be zero, contradicting  $e(L) = -1$ .

The present paper is part of a larger program whose aim is to study global invertibility and related questions in the complex-analytic setting, using geometric, analytic and topological tools ([3], [8], [9], [14]).

## 2 Proof of the main result.

Theorem 1.1 admits a slightly stronger formulation, which we will state and prove in this section. Replacing  $F$  by  $F - F(p)$ , where  $F(p) = q$ , we may assume  $q = 0$ .

**Theorem 2.1.** *Let  $X$  be a connected complex  $n$ -manifold that carries a complete Kähler metric of negative holomorphic sectional curvature,  $n \geq 2$ ,  $F : X \rightarrow \mathbb{C}^n$  a local biholomorphism,  $0 \in F(X)$ . Assume the existence of a subspace  $V \subset \mathbb{C}^n$ ,  $\dim V = 2$ , and of an integer  $k \geq 1$  such that:*

- a)  $F^{-1}(V)$  is connected.*
- b) For all but finitely many complex lines  $l \subset V$  that pass through 0,  $F^{-1}(l)$  is 1-connected. If  $l$  is any of these exceptional lines,  $F^{-1}(l)$  has at most  $k$  components.*

*Then the fiber  $F^{-1}(0)$  has at most  $k$  elements.*

We now proceed to show how Theorem 1.1 follows from Theorem 2.1. Let  $V \subset \mathbb{C}^n$  be a two-dimensional subspace, as in the statement of Theorem 1.1, with the property that its image in projective space intersects  $\Sigma$  (notation as in the Introduction) on a finite set. Since there are only finitely many one-dimensional subspaces  $l \subset V$  for which  $F^{-1}(l)$  is not 1-connected, hypothesis c) of Theorem 1.1 implies the existence of  $k \geq 1$  satisfying b) in Theorem 2.1. It is now clear that Theorem 2.1 implies Theorem 1.1 if one can establish that  $F^{-1}(V)$  is connected.

To this end, let us argue by contradiction and write  $F^{-1}(V) = U_1 \cup U_2$ , where  $U_1, U_2$  are non-empty open sets such that  $U_1 \cap U_2 = \emptyset$ .

The space of lines in  $V$  that pass through  $q = 0$  can be naturally identified with the projective plane  $\mathbb{CP}^1$ . If  $\pi$  is the natural projection, let  $E = \pi(V - \{0\}) \cap \Sigma \subset \mathbb{CP}^1$  be the finite (perhaps empty) set corresponding to the lines whose pre-images are not 1-connected. As the pre-image  $F^{-1}(l)$  of a complex line  $l$  is connected when  $\pi(l) \in \mathbb{CP}^1 - E$ , one has

$$\{\pi(l) \in \mathbb{CP}^1 - E : F^{-1}(l) \cap U_j \neq \emptyset\} = \{\pi(l) \in \mathbb{CP}^1 - E : F^{-1}(l) \subset U_j\}.$$

For  $j = 1, 2$ , these sets are non-empty, open, disjoint, and their union is  $\mathbb{CP}^1 - E$ . Since  $E$  is finite,  $\mathbb{CP}^1 - E$  remains connected, and we have a contradiction. Thus, as claimed,  $F^{-1}(V)$  is connected, finishing the proof that Theorem 1.1 follows from Theorem 2.1.  $\square$

The proof of Theorem 2.1 itself is fairly involved, and will take the remainder of this section. We start by explaining why it suffices to consider only the case  $\dim_{\mathbb{C}} X = 2$ .

Notice that, by b) of Theorem 2.1,  $F^{-1}(0)$  is contained in a single component of  $F^{-1}(V)$ . Since the latter is connected, by a), one sees that

$$F^{-1}(0) = (F|_V)^{-1}(0). \tag{2.1}$$

The metric induced from  $X$  makes  $F^{-1}(V)$  into a connected complete Kähler surface of negative holomorphic sectional curvature ([6], p.164, 176). In view of (2.1), we may

therefore assume in Theorem 2.1 that  $X = F^{-1}(V)$ , and  $F : X \rightarrow \mathbb{C}^2$  is a local biholomorphism satisfying b) of Theorem 2.1. In this context,  $V = \mathbb{C}^2$  so that a) is trivially satisfied. Furthermore,  $\Sigma \subset \mathbb{CP}^1$  is a finite set.

After the above reduction to the case  $\dim_{\mathbb{C}} X = 2$ , we argue by contradiction and assume that  $F : X \rightarrow \mathbb{C}^2$  satisfies  $\#F^{-1}(0) > k \geq 1$ .

In particular, one can choose distinct points  $p_1, p_2 \in F^{-1}(0)$ . For any  $l \in \mathbb{CP}^1 - \Sigma$ , since  $F^{-1}(l)$  is properly embedded, it follows from ([6], p. 164, 176), that the holomorphic curve  $F^{-1}(l)$  can be viewed as a complete connected real surface of negative curvature, relative to the metric induced from the complete metric of negative holomorphic sectional curvature on  $X$ . On the other hand, by assumption,  $F^{-1}(l)$  is simply-connected. Hence, at every point of  $F^{-1}(l)$ , the exponential map is a diffeomorphism by the classical Cartan-Hadamard theorem ([10], 162-163). In particular, any two points in  $F^{-1}(l)$  can be joined by a *unique* geodesic.

Again, for  $l \in \mathbb{CP}^1 - \Sigma$ , let  $\omega_l : [0, 1] \rightarrow F^{-1}(l)$  be the (unique) geodesic such that  $\omega_l(0) = p_1, \omega_l(1) = p_2$ . It is easy to see that  $\omega'_l(0)$  depends continuously on  $l$ . This is so because geodesics converge to geodesics in the  $C^2$  topology, together with the fact that there is only one geodesic in  $F^{-1}(l)$  joining  $p_1$  to  $p_2$ .

Consider the tautological line bundle over  $\mathbb{CP}^1$ , whose total space is

$$L = \{(l, v) | l \in \mathbb{CP}^1, v \in l\} \subset \mathbb{CP}^1 \times \mathbb{C}^2. \quad (2.2)$$

It is a well-known fact that the Euler number of  $L$  satisfies  $e(L) = -1 \neq 0$ .

The map

$$s : \mathbb{CP}^1 - \Sigma \rightarrow L, \quad l \mapsto dF(\omega'_l(0)), \quad (2.3)$$

will play a central role in our proof. Notice that, since  $dF(p)$  is everywhere non-singular,  $s$  is a nowhere-zero continuous section of the restriction of the tautological line bundle to the complement in  $\mathbb{CP}^1$  of the finite set  $\Sigma$ . Alternatively, we can view  $s$  as a section of  $L$  with (finitely many) singularities at the points of  $\Sigma$ .

Our strategy to prove Theorem 2.1 consists in showing that, in the presence of the condition  $\#F^{-1}(0) > k \geq 1$ , the local index of  $s$  at each one of these singularities is zero. Then, by Poincaré–Hopf index theorem ([2], p. 123-124), the Euler number of  $L$  would be zero, contradicting  $e(L) = -1$ . Thus, as claimed in the theorem, one would have  $\#F^{-1}(0) \leq k$ .

For a fixed complex line  $l_\alpha \in \Sigma$ , choose a sufficient small neighborhood  $D^2$  of  $l_\alpha$  in  $\mathbb{CP}^1$  such that  $D^2$  is diffeomorphic to an open ball in  $\mathbb{R}^2$ , and the restriction of  $L$  to  $D^2$  is a trivial bundle. The associated circle bundle to  $L$ , denoted by  $SL$ , is again a trivial bundle when restricted to  $D^2$ .

As an open subset of  $\mathbb{CP}^1$ ,  $D^2$  is also oriented. Choose the orientation on the circle  $S^1$  in such a way that the isomorphism of circle bundles  $SL|_{D^2} \simeq D^2 \times S^1$  is orientation-preserving, where  $D^2 \times S^1$  is given the product orientation.

Then, the local index of the section  $s$  at the singularity  $l_\alpha$  is the degree of the composite map:

$$\Phi : \partial \bar{D}^2 \rightarrow SL_{|\bar{D}^2} \simeq \bar{D}^2 \times S^1 \rightarrow S^1, l \mapsto \frac{s(l)}{\|s(l)\|} \mapsto \pi\left(\frac{s(l)}{\|s(l)\|}\right), \quad (2.4)$$

where  $\bar{D}^2$  is the closure of  $D^2$ ,  $\partial \bar{D}^2$  is the boundary of  $\bar{D}^2$  and  $\pi : \bar{D}^2 \times S^1 \rightarrow S^1$  is the projection. Here,  $\|s(l)\|^2 = g_0(s(l), s(l))$ , where  $g_0$  is the standard inner product on  $\mathbb{C}^2$ .

On the other hand, since we are assuming  $\#F^{-1}(0) > k \geq 1$ , and  $F^{-1}(l_\alpha)$  has at most  $k$  connected components, there must exist some component  $M^2$  of  $F^{-1}(l_\alpha)$  that contains at least two different points  $p_3, p_4 \in F^{-1}(0)$ .

Notice that  $p_1, p_2, p_3, p_4$  are not necessarily pairwise distinct, but  $p_1 \neq p_2$ , and  $p_3 \neq p_4$ . One should also observe that, although  $p_1$  and  $p_2$  have been chosen at the very beginning of the proof, the choices of  $p_3 = p_3^\alpha$  and  $p_4 = p_4^\alpha$  depend, in principle, on the particular exceptional line  $l_\alpha$ .

Despite the fact that the components of  $F^{-1}(l_\alpha)$  are not assumed to be simply-connected in Theorem 2.1, one still has uniqueness of geodesics joining  $p_3$  to  $p_4$ . This is the content of the next lemma. Without it, Theorem 2.1 would be slightly weaker. The conclusion would still be the same, but condition c) would have to be strengthened to require that each connected component of the pre-images of the exceptional lines is simply-connected.

**Lemma 2.2.** *There is a unique geodesic  $v_{l_\alpha} : [0, 1] \rightarrow M^2 \subset F^{-1}(l_\alpha)$  joining  $p_3$  to  $p_4$ .*

*Proof.* By the completeness of  $M^2$ , there is at least one such geodesic. Suppose that  $\gamma_1, \gamma_2 : [0, 1] \rightarrow M^2$  are two geodesics with  $\gamma_1(0) = \gamma_2(0) = p_3$  and  $\gamma_1(1) = \gamma_2(1) = p_4$ . Let  $U \subset M^2$  be a bounded open set containing  $\gamma_1([0, 1])$  and  $\gamma_2([0, 1])$ . For each  $x \in U$ , consider the geodesics  $\Gamma : (-\epsilon, \epsilon) \rightarrow X$  with  $\Gamma(0) = x$  and  $\Gamma'(0) \perp T_x M^2$ . Let  $\Sigma_x$  be the local surface generated by these local geodesics  $\Gamma$ . In particular,  $\Sigma_x$  intersects  $M^2$  transversally at  $x$ . It follows that for every complex line  $l \in \mathbb{CP}^1$  sufficiently close to  $l_\alpha$ ,  $\Sigma_x \cap F^{-1}(l)$  is a single point. In fact, for every  $x \in \bar{U}$ , there is a neighborhood  $U_x$  of  $x$  in  $M^2$ , and a neighborhood  $W_x$  of  $l_\alpha$  in  $\mathbb{CP}^1$  such that, for every  $y \in U_x$  and  $l \in W_x$ , one has that  $\Sigma_y \cap F^{-1}(l)$  is a single point.

Since  $\bar{U}$  is compact, it can be covered by finitely many open sets  $U_x$ . Letting  $W$  be the intersection of the (finitely many) corresponding sets  $W_x$ , we can then define a map  $\pi_{U,l}$ , for each  $l \in W$ , as follows:

$$\pi_{U,l} : U \rightarrow F^{-1}(l), x \mapsto \Sigma_x \cap F^{-1}(l). \quad (2.5)$$

Furthermore, shrinking  $W$  if necessary,  $\pi_{U,l}$  is an immersion. We also remark that the map  $\pi_{U,l_\alpha} : U \rightarrow F^{-1}(l_\alpha)$  is just the inclusion. For  $l \in W$ , consider  $F^{-1}(l)$  with its induced metric from  $X$ , and denote by  $g(l)$  the metric on  $U$  induced by the immersion  $\pi_{U,l}$ .

Denote by  $\exp_{p_3}^{g(l)}$  the exponential map of the metric  $g(l)$ , based at  $p_3$ . Since  $F$  is a local biholomorphism, for any finite positive integer  $k$  the submanifold  $F^{-1}(l)$  converges to  $F^{-1}(l_\alpha)$ ,  $C^k$ -uniformly over compact subsets of  $X$ , as the complex line  $l$  tends to  $l_\alpha$ . In particular, taking  $W$  smaller if necessary, we can assume that the domain of  $\exp_{p_3}^{g(l)}$  contains  $\gamma'_1(0)$  and  $\gamma'_2(0)$  for all  $l \in W$ .

Given a curve  $(-\epsilon, \epsilon) \rightarrow W$ ,  $t \mapsto l(t)$ ,  $l(0) = l_\alpha$ , we want to find a smooth curve  $t \mapsto \xi(t)$  in  $T_{p_3}X$ , such that

$$H(t, \xi(t)) := \exp_{p_3}^{g(l(t))}(\xi(t)) = p_4, \quad \xi(0) = \gamma'_1(0). \quad (2.6)$$

That such a curve  $\xi(t)$  exists follows from the implicit function theorem. Indeed, one only needs to check that  $\frac{\partial H}{\partial \xi}|_{t=0} = d \exp_{p_3}^{g(l_\alpha)}$  is invertible, and this is the case as  $M$  has non-positive curvature.

As a consequence of (2.6), for small enough  $t \neq 0$  there is a geodesic in the Riemannian manifold  $(U, g(t))$  joining  $p_3$  to  $p_4$ , with initial vector close to  $\gamma'_1(0)$ . Since the metric  $g(t)$  is induced by the immersion  $\pi_{U,l} : U \rightarrow F^{-1}(l)$ , and the points  $p_3$  and  $p_4$  are kept fixed by  $\pi_{U,l} : U \rightarrow F^{-1}(l)$ , it follows that, likewise, there is a geodesic in  $F^{-1}(l(t))$  that joins  $p_3$  to  $p_4$ , and whose tangent vector can be made as close to  $\gamma'_1(0)$  as one wants by taking  $t > 0$  sufficiently small.

The preceding discussion applies to the geodesic  $\gamma_2$  as well. Hence, for  $t > 0$  sufficiently close to zero, there are geodesics in  $F^{-1}(l(t))$  that join  $p_3$  to  $p_4$ , and whose initial vectors converge to  $\gamma'_1(0)$  and  $\gamma'_2(0)$  in  $T_{p_3}X$ , respectively, as  $t \rightarrow 0$ .

But  $F^{-1}(l)$  is 1-connected for  $l \neq l_\alpha$ , and so there is only one geodesic in  $F^{-1}(l)$  joining  $p_3$  and  $p_4$ . In particular, from the above paragraph one must have  $\gamma'_1(0) = \gamma'_2(0)$ , so that the geodesics  $\gamma_1$  and  $\gamma_2$  are the same. This concludes the proof of Lemma 2.2.  $\square$

By taking the disc  $D^2$  used in the definition of  $\Phi$  to be small enough, we can assume that  $D^2 \cap \Sigma = \{l_\alpha\}$ . Then, for any  $l \in D^2$ ,  $l \neq l_\alpha$ ,  $F^{-1}(l)$  is 1-connected.

Hence,  $p_3, p_4 \in F^{-1}(l)$  and there is a unique geodesic  $v_l : [0, 1] \rightarrow F^{-1}(l)$  such that  $v_l(0) = p_3, v_l(1) = p_4$ . Let  $\Psi$  be the composite map

$$\Psi : \partial \bar{D}^2 \rightarrow SL_{|\bar{D}^2} \simeq \bar{D}^2 \times S^1 \rightarrow S^1, \quad l \mapsto \frac{\sigma(l)}{\|\sigma(l)\|} \mapsto \pi\left(\frac{\sigma(l)}{\|\sigma(l)\|}\right), \quad (2.7)$$

where  $v'_l(0) = \frac{d}{dt}v_l(t)|_{t=0}$ ,  $\pi$  is the projection onto the second factor, and

$$\sigma : \partial \bar{D}^2 \rightarrow L, \quad l \mapsto dF(v'_l(0)). \quad (2.8)$$

Notice the similarity between the definitions of the maps  $\Phi$  and  $\Psi$ : in the first instance the geodesics join  $p_1$  to  $p_2$ , while in the second they join  $p_3$  to  $p_4$ .

By Lemma 2.2, we can define  $\bar{\sigma} : \bar{D}^2 \rightarrow L$  by

$$l_\alpha \mapsto dF(v'_{l_\alpha}(0)), \quad \text{and } l \mapsto dF(v'_l(0)) \text{ if } l \neq l_\alpha. \quad (2.9)$$

A simple argument using the uniqueness of  $v_{l_\alpha}$  shows that  $\bar{\sigma}$  is a continuous extension of  $\sigma$ . Accordingly,  $\Psi$  extends to the composite map

$$\bar{\Psi} : \bar{D}^2 \rightarrow SL_{|\bar{D}^2} \simeq \bar{D}^2 \times S^1 \rightarrow S^1, \quad l \mapsto \frac{\bar{\sigma}(l)}{\|\bar{\sigma}(l)\|} \mapsto \pi\left(\frac{\bar{\sigma}(l)}{\|\bar{\sigma}(l)\|}\right). \quad (2.10)$$

**Lemma 2.3.** *Denote by  $g$  the metric of  $X$ . Then, there exists a compact set  $K \subset X$  and disjoint neighborhoods  $U_\alpha$  in  $\mathbb{CP}^1$  of the finitely many exceptional lines  $l_\alpha$  such that, for every  $\epsilon > 0$ , there exists a metric  $\hat{g}$  on  $X$  satisfying*

- i)  $\text{supp}(g - \hat{g}) \subset K$ ,  $\{p_1, p_2, p_3^\alpha, p_4^\alpha\} \cap K = \emptyset$ .
- ii)  $\|g - \hat{g}\|_{C^2} \leq \epsilon$ .
- iii) *For any  $l \in U_\alpha$ , no three points in the set  $\{p_1, p_2, p_3^\alpha, p_4^\alpha\}$  lie on the same geodesic of  $F^{-1}(l)$ , relative to the metric induced from  $\hat{g}$ .*

*Proof.* Since geodesics converge to geodesics, once iii) is established for the exceptional lines  $l_\alpha$  themselves, the existence of  $U_\alpha$  follows.

Suppose, for instance, that for some  $l_\alpha$  fixed, the oriented geodesic on  $F^{-1}(l_\alpha)$  joining  $p_1$  and  $p_2$  also contains  $p_3 = p_3^\alpha$ , and  $p_3 > p_2$  (in the obvious sense).

Let  $q$  be a point in the open geodesic segment  $(p_2, p_3)$  that does not belong to the (discrete) set  $F^{-1}(0)$ . Choose a small geodesic ball  $B_\epsilon(q)$  in the ambient manifold  $X$ , and set  $A = B_\epsilon(q) \cap F^{-1}(l_\alpha)$ .

For  $p \in \partial A$ , denote by  $\nu(p)$  the set of unit tangent vectors at  $p$  corresponding to the geodesics in  $F^{-1}(l_\alpha)$  joining  $p$  to  $p_3$ . Since the curvature is negative, it follows from the arguments in Lemma 2.2 that the set  $\nu(p)$  actually consists of a single point if  $\epsilon$  is small enough, and the map  $\nu : A \rightarrow T\partial A$ ,  $p \mapsto \nu(p)$ , is continuous.

It is now clear that there is a metric  $\tilde{g}$  on  $F^{-1}(l_\alpha)$  for which  $\tilde{g} - g|_{F^{-1}(l_\alpha)}$  is supported inside  $A$ ,  $\|\tilde{g} - g|_{F^{-1}(l_\alpha)}\|_{C^2}$  is as small as desired and, furthermore, the following holds:

(†) The  $\tilde{g}$ -geodesic whose initial point is the first point in  $\partial A$  of the continuation of the  $g$ -geodesic  $[p_1, p_2]$  hits the boundary  $\partial A$  again at some point  $p$  with a tangent vector that is not a multiple of  $\nu(p)$ .

We omit the details that (†) can always be achieved, but point out an additional simplification. By working with geodesic polar coordinates one can first make a small compactly supported perturbation of the original metric in  $F^{-1}(l_\alpha)$  near  $q$ , and assume in (†) that the curvature is a negative constant in a neighborhood of  $q$ .

To continue with the proof, it follows from (†) and the definition of  $\nu$  that the  $\tilde{g}$ -geodesic that passes through  $p_1$  and  $p_2$  does not contain  $p_3$ . Since  $\tilde{g}$  is an arbitrarily small perturbation of the metric induced by  $g$ , and these metrics coincide off  $B_\epsilon(q)$ , it is possible to arrange matters so that there is a small perturbation  $\hat{g}$  of the ambient metric  $g$  that coincides with  $g$  off  $B_\epsilon(q)$  and induces  $\tilde{g}$  on  $F^{-1}(l_\alpha)$ .

Since there are only finitely many exceptional lines  $l_\alpha$ , and finitely many possibilities for three of the points in  $\{p_1, p_2, p_3^\alpha, p_4^\alpha\}$  to lie on the same geodesics of  $F^{-1}(l_\alpha)$ , one can



choose the points  $q$  in the discussion above to be pairwise distinct for the various choices of  $l_\alpha$ , and  $\epsilon$  to be small enough, so that the corresponding balls  $B_\epsilon(q)$  are disjoint.

Hence, by making a finite number of arbitrarily small perturbations of  $g$ , all supported on arbitrarily small disjoint sets away from  $F^{-1}(0)$ , we can construct a metric  $\hat{g}$  as in the statement of the lemma by taking  $K$  to be the union of the various closed disjoint open balls  $\overline{B_\epsilon}$ .

Notice that  $\hat{g}$  is not Kähler, but by taking  $\epsilon$  small enough the metric induced by  $\hat{g}$  on any  $F^{-1}(l)$  has negative curvature.  $\square$

**Proposition 2.4.** *The maps  $\Phi$  and  $\Psi$  are homotopic.*

The proof of this proposition is divided into three cases which need to be examined separately.

**Case 1:**  $p_1 = p_3$  and  $p_2 = p_4$ .

This is the trivial alternative, since  $\Phi = \Psi$ .

**Case 2:**  $p_1 = p_3$  and  $p_2 \neq p_4$ , or  $p_2 = p_4$  and  $p_1 \neq p_3$ .

We shall assume that  $p_1 = p_3$ ,  $p_2 \neq p_4$ . The proof is similar if  $p_2 = p_4$ ,  $p_1 \neq p_3$ . Let  $\rho_l : [0, 1] \rightarrow F^{-1}(l)$  be the unique (non-constant) geodesic such that  $\rho_l(0) = p_2$ ,  $\rho_l(1) = p_4$ , as depicted in figure 1.

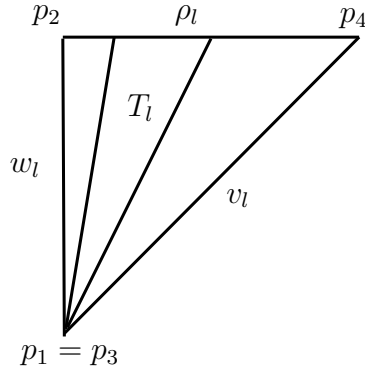


Figure 1

By Lemma 2.3, we can choose  $D^2$  in the definition of  $\Psi$  small enough so that  $p_1$  does not lie in  $\rho_l$  for any  $l \in \bar{D}^2$ . Let  $T_l : [0, 1] \times [0, 1] \rightarrow F^{-1}(l)$  be given by the family of geodesics  $T_l(s, \cdot)$  such that  $T_l(s, 0) = p_1$ ,  $T_l(s, 1) = \rho_l(s)$ , where  $l \in \partial \bar{D}^2$ .

Recall that  $\omega_l : [0, 1] \rightarrow F^{-1}(l)$  is the unique geodesic such that  $\omega_l(0) = p_1$ ,  $\omega_l(1) = p_2$ , and  $v_l : [0, 1] \rightarrow F^{-1}(l)$  is the unique geodesic such that  $v_l(0) = p_3$ ,  $v_l(1) = p_4$ . Thus,  $s \mapsto T_l(s, \cdot)$  is a homotopy between  $\omega_l$  and  $v_l$ . Since  $p_1$  does not lie in  $\rho_l$ , by Lemma 2.3, we see that

$$T'_l(s, 0) = \frac{\partial}{\partial t} T_l(s, t)|_{t=0} \neq 0$$

for any  $l \in \partial \bar{D}^2$  and  $s \in [0, 1]$ . Let  $\Theta$  be the composite map

$$\partial \bar{D}^2 \times [0, 1] \rightarrow SL_{|\bar{D}^2} \simeq \bar{D}^2 \times S^1 \rightarrow S^1, \quad (l, s) \mapsto \frac{\tau(l, s)}{\|\tau(l, s)\|} \mapsto \pi\left(\frac{\tau(l, s)}{\|\tau(l, s)\|}\right), \quad (2.11)$$

where

$$\tau : \partial \bar{D}^2 \times [0, 1] \rightarrow L, \quad (l, s) \mapsto dF(T'_l(s, 0)). \quad (2.12)$$

It is now easy to see that the map  $\Theta$  is a homotopy between  $\Phi$  and  $\Psi$ .

**Case 3:**  $p_1 \neq p_3$  and  $p_2 \neq p_4$ .

By Lemma 2.3, we can choose  $D^2$  small enough such that for any  $l \in \partial \bar{D}^2$ , no three points of  $p_1, p_2, p_3, p_4$  lie in the same geodesic in  $F^{-1}(l)$ . Fix  $l \in \partial \bar{D}^2$  and let  $\alpha_l(s) : [0, 1] \rightarrow F^{-1}(l)$  be the unique geodesic such that  $\alpha_l(0) = p_1, \alpha_l(1) = p_3$ . Likewise, let  $\beta_l(s) : [0, 1] \rightarrow F^{-1}(l)$  be the unique geodesic such that  $\beta_l(0) = p_2, \beta_l(1) = p_4$ .

Let  $H_l : [0, 1] \times [0, 1] \rightarrow F^{-1}(l)$  be given by the family of geodesics joining  $H_l(s, 0) = \alpha_l(s)$  to  $H_l(s, 1) = \beta_l(s)$ . Notice that, for each  $s \in [0, 1]$ ,  $H_l(s, \cdot)$  is uniquely determined by  $\alpha_l(s)$  and  $\beta_l(s)$ , since  $F^{-1}(l)$  is a Cartan-Hadamard surface. Then  $H_l$  induces a homotopy between  $\omega_l$  and  $v_l$  (see figure 2).

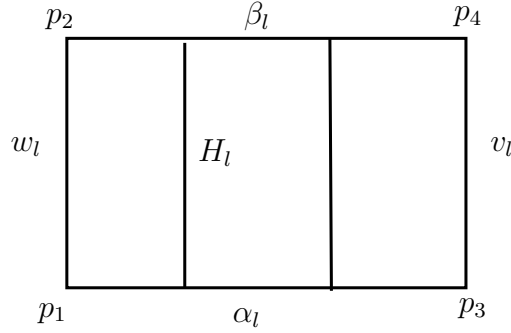


Figure 2

**Lemma 2.5.** *After labeling  $p_3, p_4$  suitably,  $\alpha_l$  does not intersect  $\beta_l$  for any  $l \in \partial \bar{D}^2$ . In particular, for any  $l \in \partial \bar{D}^2$  and  $s \in [0, 1]$ ,  $H'_l(s, 0) = \frac{\partial}{\partial t} H_l(s, t)|_{t=0} \neq 0$ .*

We need the following well-known convexity result in the geometry of surfaces of non-positive curvature. For completeness, we include its proof.

**Lemma 2.6.** *Let  $N^2$  be a Cartan-Hadamard surface and  $\delta : (-\infty, +\infty) \rightarrow N^2$  a geodesic in  $N^2$ . Then each component of  $N^2 - \delta$  is convex.*

*Proof.* Fix any two points  $x_1, x_2$  lying in the same component  $U$  of  $N^2 - \delta$ . Let  $\gamma : [0, 1] \rightarrow N^2$  be the unique geodesic such that  $\gamma(0) = x_1$  and  $\gamma(1) = x_2$ , as shown in figure 3.

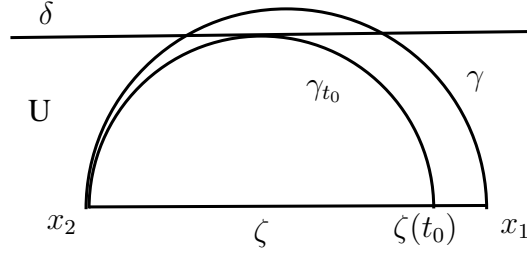


Figure 3

Assume, by contradiction, that  $\gamma$  intersects  $\delta$ . Take any continuous curve  $\zeta : [0, 1] \rightarrow N^2$  such that  $\zeta(0) = x_1, \zeta(1) = x_2$ , and  $\zeta$  does *not* intersect  $\delta$ . Consider

$$t_0 = \sup\{t | \gamma_t \cap \delta \neq \emptyset, t \in [0, 1]\},$$

where  $\gamma_t : [0, 1] \rightarrow N^2$  is the unique geodesic such that  $\gamma_t(0) = \zeta(t)$  and  $\gamma_t(1) = x_2$  for any  $t \in [0, 1]$ . Notice that  $\gamma_0 = \gamma$ . Since  $\zeta$  does not intersect  $\delta$ , we see that  $t_0 < 1$ . Then  $\gamma_{t_0}$  must be tangent to  $\delta$ , otherwise  $\gamma_t$  will also intersect  $\delta$  for some  $t > t_0$ , which contradicts the definition of  $t_0$ . Hence  $\gamma_{t_0}$  must be tangent to  $\delta$ , and so these curves must coincide as they are geodesics. Since  $x_2 \in \gamma_{t_0}$ , we have  $x_2 \in \delta$ , a contradiction.  $\square$

Now, we are going to prove Lemma 2.5. We start by fixing  $l_0 \in \partial \bar{D}^2$ . We first show that, by relabelling  $p_3$  and  $p_4$  if necessary,  $\alpha_{l_0}$  does not intersect  $\beta_{l_0}$ . Suppose  $\alpha_{l_0} \cap \beta_{l_0} \neq \emptyset$ . By Lemma 2.3,  $p_2, p_4$  do not lie in  $\alpha_{l_0}$ . Since  $\alpha_{l_0} \cap \beta_{l_0} \neq \emptyset$ , Lemma 2.6 implies that  $p_2, p_4$  must lie in different components of  $N^2 - \alpha_{l_0}$  (see figure 4).

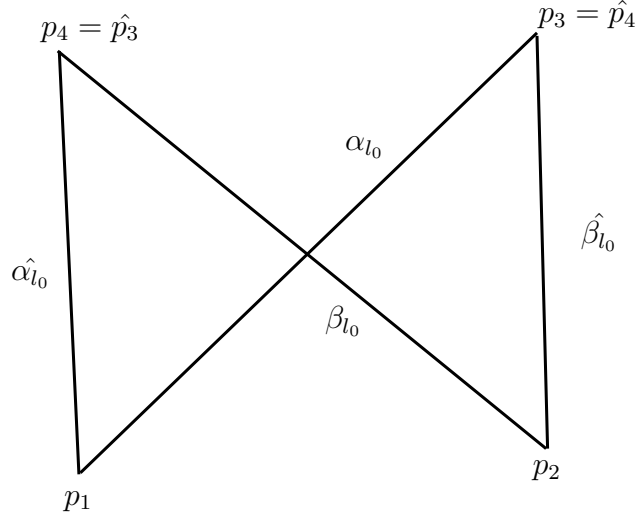


Figure 4

In this case, we interchange the labeling of  $p_3$  and  $p_4$ . In other words, set  $\hat{p}_1 = p_1$ ,  $\hat{p}_2 = p_2$ ,  $\hat{p}_3 = p_4$ ,  $\hat{p}_4 = p_3$  (figure 4). Also let  $\hat{\alpha}_{l_0} : [0, 1] \rightarrow F^{-1}(l)$  be the unique geodesic such that  $\hat{\alpha}_{l_0}(0) = p_1$ ,  $\hat{\alpha}_{l_0}(1) = \hat{p}_3$  and  $\hat{\beta}_{l_0} : [0, 1] \rightarrow F^{-1}(l)$  be the unique geodesic such

that  $\hat{\beta}_{l_0}(0) = p_2$ ,  $\hat{\beta}_{l_0}(1) = \hat{p}_4$ . By Lemma 2.6,  $\hat{\alpha}_{l_0}$  and  $\hat{\beta}_{l_0}$  lie in different components of  $N^2 - \alpha_{l_0}$ , and hence do not intersect each other.

As we shall see below, the crucial point of the proof that  $\alpha_l$  does not intersect  $\beta_l$  for any  $l \in \partial\bar{D}^2$  is the fact that, as it was arranged above, after a possible relabeling of  $p_3$  and  $p_4$ , the conclusion holds for a single line  $l_0 \in \partial\bar{D}^2$ . We now proceed to show that, in fact,  $\alpha_l \cap \beta_l = \emptyset$  for all  $l \in \partial\bar{D}^2$ .

By continuity,  $\alpha_l$  does not intersect  $\beta_l$  if  $l$  is sufficiently close to  $l_0$ . Suppose that, for some  $l$ ,  $\alpha_l$  does indeed intersect  $\beta_l$ . We are going to derive a contradiction.

Let  $\ell$  be the complex line such that  $\alpha_\ell$  intersects  $\beta_\ell$  for the the “first time”, as a variable line in  $\partial\bar{D}^2$  that started at  $l_0$  moves towards  $\ell$  (say, in the positive sense in  $\partial\bar{D}^2$ ).

Considering what happens when the first contact occurs, one sees that one of the possibilities below must occur (see figures 5, 6 and 7):

- i)  $\alpha_\ell$  is tangent to  $\beta_\ell$  at some interior point of  $\beta_\ell$ .
- ii)  $\alpha_\ell \cap \beta_\ell = \{\beta_\ell(0)\} = \{p_2\}$ .
- iii)  $\alpha_\ell \cap \beta_\ell = \{\beta_\ell(1)\} = \{p_4\}$ .

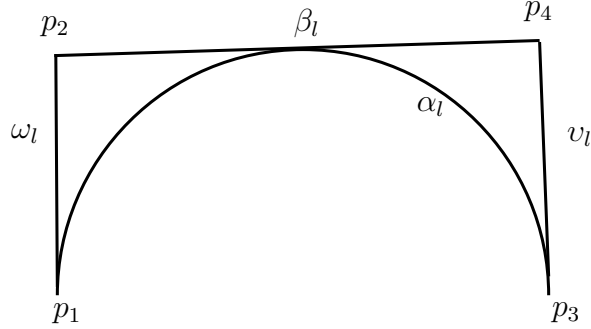


Figure 5

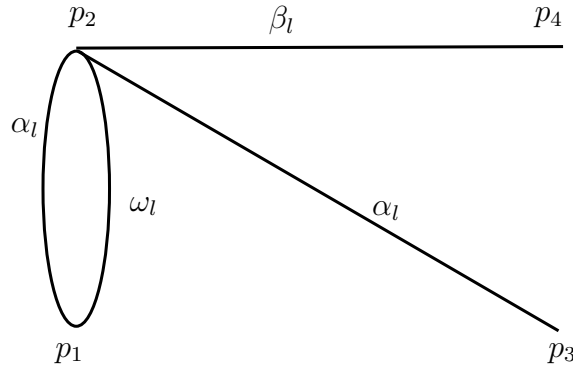


Figure 6

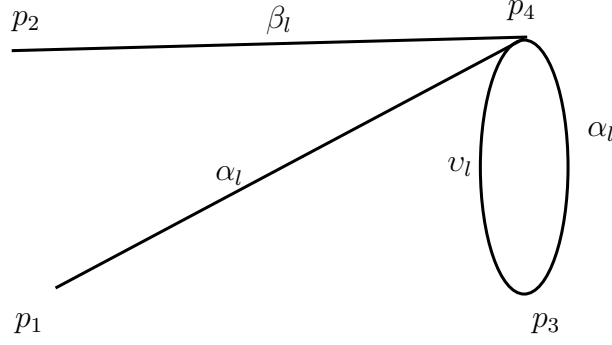


Figure 7

In alternative i),  $\alpha_\ell$  is tangent to  $\beta_\ell$ , and so these curves must coincide since they are geodesics. Then  $p_1, p_2, p_3, p_4$  lie in the same geodesic in  $F^{-1}(\ell)$ , contradicting Lemma 2.3.

If ii) holds,  $\alpha_\ell \cap \beta_\ell = \beta_\ell(0) = p_2$ , and so  $\alpha_\ell$  passes through both  $p_1$  and  $p_2$ . Recall that  $\omega_\ell : [0, 1] \rightarrow F^{-1}(\ell)$  is the unique geodesic such that  $\omega_\ell(0) = p_1$  and  $\omega_\ell(1) = p_2$ . Since  $F^{-1}(\ell)$  is a Hadamard surface, we see that  $\alpha_\ell$  must coincide with  $\omega_\ell$ . Since  $\alpha_\ell(1) = p_3$ , we see that  $p_1, p_2, p_3$  lie in the same geodesic  $\omega_\ell$ , which is again a contradiction to Lemma 2.3. For the same reason, iii) cannot happen either.

Hence,  $\alpha_l$  does not intersect  $\beta_l$  for any  $l \in \partial \bar{D}^2$ , thus establishing the first half of Lemma 2.5.

Recall that  $H_l(s, t) : [0, 1] \times [0, 1] \rightarrow F^{-1}(l)$  provides a family of geodesics  $H_l(s, \cdot)$  such that

$$H_l(s, 0) = \alpha_l(s), \quad H_l(s, 1) = \beta_l(s).$$

Since  $\alpha_l(s) \neq \beta_l(s)$  for all  $s \in [0, 1]$ , we see that the connecting geodesic is non-constant, and so

$$H'_l(s, 0) = \frac{\partial}{\partial t} H(s, t)|_{t=0} \neq 0 \tag{2.13}$$

for any  $l \in \partial \bar{D}^2$  and  $s \in [0, 1]$ . This concludes the proof of Lemma 2.5.  $\square$

We can now complete the proof of Proposition 2.4. Let  $\Lambda$  be the composite map

$$\partial \bar{D}^2 \times [0, 1] \rightarrow SL_{|\bar{D}^2} \simeq \bar{D}^2 \times S^1 \rightarrow S^1, \quad (l, s) \mapsto \frac{\varsigma(l, s)}{\|\varsigma(l, s)\|} \mapsto \pi\left(\frac{\varsigma(l, s)}{\|\varsigma(l, s)\|}\right), \tag{2.14}$$

where

$$\varsigma : \partial \bar{D}^2 \times [0, 1] \rightarrow L, \quad (l, s) \mapsto dF(H'_l(s, 0)). \tag{2.15}$$

It follows from the properties of  $H$  that  $\Lambda$  is a homotopy between  $\Phi$  and  $\Psi$ . This completes the discussion of Case 3, thus proving Proposition 2.4.  $\square$

We can now finish the proof of Theorem 2.1. For each exceptional line  $l_\alpha$  we constructed maps  $\Phi, \Psi : \partial \bar{D}^2 \rightarrow S^1$ . By Proposition 2.4,  $\deg(\Phi) = \deg(\Psi)$ . Since  $\Psi$  extends to a continuous map  $\bar{\Psi} : \bar{D}^2 \rightarrow S^1$ ,  $\deg(\Psi) = 0$  ([4], p. 126).

Hence  $\deg(\Phi) = 0$  and therefore, as it was explained prior to the definition of  $\Phi$ , the local index of  $s$  at  $l_\alpha$  is zero. Summing over  $\alpha$ , and using the Poincaré-Hopf index theorem ([2], p. 123-124), the Euler number of the tautological line bundle  $L$  would be zero, a contradiction to  $e(L) = -1$ .

### 3 Examples.

A) It is essential in Theorem 1.1 that the pre-image of each exceptional line should have only finitely many components (hypothesis c)), otherwise the fiber may be infinite.

In order to see this, consider the simple local biholomorphism  $F : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ , given by

$$F(z_1, z_2) = (e^{z_1} - 1, z_2),$$

so that  $\#F^{-1}(0, 0) = \infty$ . If  $\lambda \in \mathbb{C}$ , the pre-image of the line  $w_2 = \lambda w_1$  is a graph over  $\mathbb{C}$ , hence 1-connected. In particular, a) and b) are satisfied.

On the other hand, the pre-image of the line  $\omega_1 = 0$  is  $\bigcup_{j=1}^{\infty} (\{2\pi j \sqrt{-1}\} \times \mathbb{C})$ . Hence, although there is only one exceptional line, its pre-image has infinitely many components.

B) The estimate in Theorem 1.1 is sharp.

We begin with a locally univalent entire function  $f : \mathbb{C} \rightarrow \mathbb{C}$  that covers 0 exactly  $k$  times,  $1 \leq k < \infty$ . These functions can be constructed as follows (we are grateful to A. Weitsman for kindly explaining to us the case  $k = 1$ ).

Let

$$P(z) = \prod_{j=1}^k (z - j),$$

so that

$$P'(j) = \prod_{1 \leq n \leq k, n \neq j} (j - n) \neq 0.$$

Let  $\psi$  be any entire function that solves the finite interpolation problem

$$\psi(j) = \log P'(j), \quad j = 1, \dots, k,$$

for some branch of the logarithm; for instance, it is easy to see that there is a (unique) solution  $\psi$  that is a polynomial of degree  $\leq k - 1$ .

Set

$$\phi(z) = (P(z))^{-1} (e^{\psi(z)} - P'(z)),$$

and observe that  $z = j$  is a removable singularity for  $\phi$ , so that  $\phi$  extends to an entire function. Let  $\Phi$  be an entire function satisfying  $\Phi' = \phi$  and define

$$f(z) = P(z)e^{\Phi(z)}.$$

Hence,  $f$  vanishes precisely at  $1, \dots, k$ , with multiplicity one at all these points. Furthermore,  $f$  is locally univalent, since

$$f'(z) = (P(z)\phi(z) + P'(z))e^{\Phi(z)} = e^{\psi(z)}e^{\Phi(z)} \neq 0.$$

Consider now the map  $F : \mathbb{C}^2 \rightarrow \mathbb{C}^2$  given by  $F(z_1, z_2) = (f(z_1), z_2)$ . The Jacobian determinant of  $F$  at  $(z_1, z_2)$  is  $f'(z_1)$ , hence non-zero. The fiber of  $(0, 0)$  has precisely  $k$  elements, namely  $(1, 0), \dots, (k, 0)$ .

As in Example A), if  $\lambda \in \mathbb{C}$  the pre-image of the line  $w_2 = \lambda w_1$  is a graph over  $\mathbb{C}$ , hence 1-connected. On the other hand, if  $l$  is the line  $w_1 = 0$ ,  $F^{-1}(l)$  is homeomorphic to

$$f^{-1}(0) \times \mathbb{C} = \bigcup_{j=1}^k (\{j\} \times \mathbb{C}),$$

which has precisely  $k$  connected components, the same number of points in the fiber  $F^{-1}(0, 0)$ . This shows that equality can be achieved in (1.1).

C) The topological hypothesis b), to the effect that that the pre-images of the generic lines passing through  $q$  should be 1-connected, is essential for the validity of Theorem 2.1.

In order to produce an example illustrating this point, we need the result that the product of Kähler manifolds of negative holomorphic sectional curvature also has these properties. Since we were unable to locate a reference for the part of this statement concerning curvature, a proof is included in an appendix.

Every metric on an orientable real surface is Kähler, relative to the natural complex structure, and so one can endow  $\mathbb{C} - \{1/2\}$  with a complete conformal metric of negative curvature (for instance, by considering the Weierstrass representation of a catenoid).

Taking the corresponding product metric on  $\Omega := (\mathbb{C} - \{1/2\}) \times (\mathbb{C} - \{1/2\})$ , and using the result in the appendix, one obtains a complete Kähler metric of negative holomorphic sectional curvature, as required in the statement of Theorem 1.1.

Alternatively, and in a less elementary way, such a metric exists because the Behnke-Stein theorem ([7], p. 240) applies to show that  $\Omega$ , being the product of two open Riemann surfaces, is also Stein. In particular,  $\Omega$  can be given the metric induced from some proper embedding  $\Omega \rightarrow \mathbb{C}^N$ , where  $\mathbb{C}^N$  is endowed with a complete Kähler metric of negative holomorphic sectional curvature [12].

Next, consider the map

$$G : \mathbb{C}^2 \rightarrow \mathbb{C}^2, \quad G(z_1, z_2) = (z_1^2 - z_1, z_2^2 - z_2).$$

The restriction  $F = G|_{\Omega} : \Omega \rightarrow \mathbb{C}^2$  is a local biholomorphism, hypothesis a) is satisfied, and the fiber  $F^{-1}(0, 0)$  consists of the four points  $(0, 0), (0, 1), (1, 0), (1, 1)$ . Giving coordinates  $(w_1, w_2)$  to  $\mathbb{C}^2$  (the target of  $F$ ), one sees that the complex lines  $l$  through  $(0, 0)$  are of the following two types:  $w_1 = 0$ , or  $w_2 = kw_1$  for an arbitrary  $k \in \mathbb{C}$ .

We now analyze when  $F^{-1}(l)$  is connected, for  $l$  of the second kind. Since  $G^{-1}(l)$  is given by the zero set of the complex polynomial

$$P_k(z_1, z_2) = z_2^2 - kz_1^2 + kz_1 - z_2,$$

and  $F^{-1}(l)$  is obtained from  $G^{-1}(l)$  by deleting the points corresponding to  $z_1 = 1/2$  or  $z_2 = 1/2$ , one sees that  $F^{-1}(l)$  is not simply-connected.

In order to show that the pre-image  $F^{-1}(l)$  of a generic line is connected, it suffices to show that the same holds for  $G^{-1}(l)$ , since the former is obtained from the latter by deleting finitely many points.

Therefore, one needs to know the values of  $k$  for which the polynomial  $P_k(z_1, z_2)$  is irreducible. The following arguments are standard in elementary algebraic geometry. If  $P_k$  is reducible, the same is true of its homogenization

$$Q_k(z_1, z_2, Z) = Z^2 P_k(z_1/Z, z_2/Z) = z_2^2 - kz_1^2 + kz_1Z - z_2Z.$$

The usual criterion for the irreducibility of conics in the projective plane, in terms of the non-vanishing of the determinant of the associated quadratic form ([11], p. 212), now applies to show that  $Q_k$  is reducible if and only if  $k = 0, 1$ .

For  $k = 0$ ,

$$F^{-1}\{w_2 = 0\} = (\{z_2 = 0\} - \{(1/2, 0)\}) \cup (\{z_2 = 1\} - \{(1/2, 1)\}).$$

For  $k = 1$ , since  $P_1(z_1, z_2) = (z_2 - z_1)(z_2 + z_1 - 1)$ ,

$$F^{-1}\{w_2 = w_1\} = (\{z_2 = z_1\} - \{(1/2, 1/2)\}) \cup (\{z_2 + z_1 = 1\} - \{(1/2, 1/2)\}).$$

The pre-image of the line  $\{w_1 = 0\}$  is

$$F^{-1}\{w_1 = 0\} = (\{z_1 = 0\} - \{(0, 1/2)\}) \cup (\{z_1 = 1\} - \{(1, 1/2)\}).$$

Hence, there are three lines with disconnected pre-images, namely:  $w_1 = 0, w_1 = w_2$ , and  $w_2 = 0$ . For each one of them the pre-image has two connected components (none of which is simply-connected, but this is not required in hypothesis c) of Theorem 1.1). For any other line (i.e.,  $w_2 = kw_1$ ,  $k \neq 0, 1$ ), the pre-image is connected but not simply-connected.

Thus, the number of connected components of the pre-image of each of the three lines with disconnected pre-image is two, which is strictly less than four, the number of points in  $F^{-1}(0, 0)$ . Hence, the estimate (1.2) does not hold.

This is compatible with Theorem 1.1. Indeed, although a) and c) are satisfied (in fact, the pre-image of any complex line has only finitely many components), the pre-image of the line  $w_2 = kw_1$ ,  $k \neq 0, 1$ , is connected but not simply-connected. Since the complement of any finite set in  $\mathbb{CP}^1$  would have to contain points corresponding to such lines, regardless of what Zariski open set one considers there will be generic lines whose pre-images are not simply-connected, and so hypothesis b) does not hold.



## 4 Appendix.

**Lemma 4.1.** *Let  $(M_i, J_i, g_i)$  be Kähler manifolds of negative holomorphic sectional curvature,  $j = 1, 2$ . Then*

$$(M_1 \times M_2, J_1 \oplus J_2, g_1 \times g_2)$$

*is also a Kähler manifold of negative holomorphic sectional curvature.*

It is standard that the product manifold is also Kähler. In order to show that  $g_1 \times g_2$  has negative holomorphic sectional curvature, let  $\nabla, \nabla^1, \nabla^2$  be the Levi-Civita connections associated to  $g_1 \times g_2, g_1, g_2$ , respectively. For  $X_1, X_2 \in \Gamma(TM_1), Y_1, Y_2 \in \Gamma(TM_2)$ , we have, for  $i, j = 1, 2$ ,

$$\nabla_{X_i} Y_j = 0, \nabla_{Y_i} X_j = 0, [X_i, Y_j] = 0, \nabla_{X_i} X_j = \nabla_{X_i}^1 X_j, \nabla_{Y_i} Y_j = \nabla_{Y_i}^2 Y_j. \quad (4.1)$$

For any  $Z \in \Gamma(TM_1 \times TM_2), Z \neq 0$ , we can write  $Z = X + Y$ , where  $X \in \Gamma(TM_1)$  and  $Y \in \Gamma(TM_2)$ . Let  $R, R_1, R_2$  be the Riemannian curvature tensors associated to  $g_1 \times g_2, g_1, g_2$ , respectively. Letting  $J = J_1 \oplus J_2$ , one has

$$\begin{aligned} R(Z, JZ, Z, JZ) &= R(X + Y, JX + JY, X + Y, JX + JY) \\ &= R(X, JX, X, JX) + R(X, JX, X, JY) + R(X, JX, Y, JX) + R(X, JX, Y, JY) \\ &\quad + R(X, JY, X, JX) + R(X, JY, X, JY) + R(X, JY, Y, JX) + R(X, JY, Y, JY) \\ &\quad + R(Y, JX, X, JX) + R(Y, JX, X, JY) + R(Y, JX, Y, JX) + R(Y, JX, Y, JY) \\ &\quad + R(Y, JY, X, JX) + R(Y, JY, X, JY) + R(Y, JY, Y, JX) + R(Y, JY, Y, JY). \end{aligned}$$

Using (4.1) and the definition of the curvature tensor, one can show that among the sixteen summands above only the first and the last terms can possibly be non-zero. Thus,

$$\begin{aligned} R(Z, JZ, Z, JZ) &= R(X, JX, X, JX) + R(Y, JY, Y, JY) \\ &= R_1(X, J_1 X, X, J_1 X) + R_2(Y, J_2 Y, Y, J_2 Y) < 0, \end{aligned}$$

as  $Z \neq 0$  implies  $X \neq 0$  or  $Y \neq 0$ , and the holomorphic sectional curvatures are negative.

The verification that the intermediate terms vanish is straightforward. We illustrate this by computing the second, fourth, and seventh terms:

$$R(X, JX, X, JY) = g(\nabla_X \nabla_{JX} X - \nabla_{JX} \nabla_X X - \nabla_{[X, JX]} X, JY)$$

$$= g(\nabla_X^1 \nabla_{JX}^1 X - \nabla_{JX}^1 \nabla_X^1 X - \nabla_{[X, JX]}^1 X, JY) = 0,$$

since  $JY$  is orthogonal to  $TM_1$ .

Similarly,

$$R(X, JX, Y, JY) = g(\nabla_X \nabla_{JX} Y - \nabla_{JX} \nabla_X Y - \nabla_{[X, JX]} Y, JY) = g(0, JY) = 0,$$

and

$$R(X, JY, Y, JX) = g(\nabla_X \nabla_{JY} Y - \nabla_{JY} \nabla_X Y - \nabla_{[X, JY]} Y, JX) = g(0, JX) = 0.$$

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